Bases for spin systems and qudits from angular momentum theory

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Abstract: Spin bases of relevance for quantum systems with cyclic symmetry as well as for quantum information and quantum computation are constructed from the theory of angular momentum. This approach is connected to the use of generalized Pauli matrices (in dimension d) arising from a polar decomposition of the group SU_2. Numerous examples are given for $d = 2, 3$ and 4.

Keywords: Spin systems, qubits, qudits, generalized Pauli matrices, mutually unbiased bases, generalized Pauli group.

1. INTRODUCTION

Hilbert spaces of finite dimension d play an important role for the quantum mechanical description of dynamical systems and systems of qudits (qubits correspond to $d = 2$, qudits to d arbitrary). Generally speaking, bases for finite-dimensional subspaces of the representation space of the group SO_3 or its universal covering SU_2 can be used for spin systems (used in spectroscopy and quantum chemistry) and qudits (used in quantum information and quantum computation). Such bases can be constructed from tools developed for a study of supersymmetric quantum mechanics [1–4] and for a nonstandard approach to the representations of SU_2 [5–7].

It is the object of this talk to describe a method for constructing bases which are of central importance in quantum information theory, namely, mutually unbiased bases (MUBs). In dimension d, two bases are said to be unbiased if and only if the modulus of the inner product of any vector of one basis with any vector of the other one is equal $1/\sqrt{d}$ [8–17]. These bases are of paramount importance for quantum cryptography and quantum state tomography.

We develop here an approach that gives a complete solution for the construction of MUBs in the case where the dimension d of the considered Hilbert space is a prime number. This approach is based on a nonstandard approach to the theory of angular momentum viewed through the prism of the Lie algebra of SU_2. The concepts of Weyl pairs and generalized Pauli group useful in quantum computation and quantum information are also briefly discussed.

Most of the material presented in this work takes its origin in papers published by the author and its collaborators [18–26].

The notation adopted here is the one used in quantum mechanics. Let us simply mention that $\delta_{a,b}$ stands for the Kronecker symbol for a and b, $A^\dagger$ denotes the adjoint of the operator A, and $[A, B]$ is the commutator of the operators A and B.

2. SPIN BASES

2.1 *Angular momentum states*

Let us consider a generalized angular momentum. We denote as j^2 and j_z its square and z-component, respectively. The common orthonormalized eigenvectors of the two commuting operators j^2 and j_z are written $|j, m\rangle$. We know that

$$j^2|j,m\rangle = j(j+1)|j,m\rangle, \quad j_z|j,m\rangle = m|j,m\rangle,$$

where $m = j, j-1, \ldots, -j$ and $2j \in \mathbf{N}$. For a fixed value of the quantum number j, we note $\mathcal{E}(2j+1) \sim \mathbf{C}^{2j+1}$ the $(2j+1)$-dimensional Hilbert space spanned by the basis

$$b_s = \{|j,m\rangle : m = j, j-1, \ldots, -j\}.$$

The basis b_s is adapted to spherical symmetry (adapted to the group SO_3 if $j \in \mathbf{N}$ or SU_2 if $2j \in \mathbf{N}$).

In the applications to spectroscopy, the generalized angular momentum can be an orbital angular momentum, a spin angular momentum, a total (spin + orbital) angular momentum, a nucleus spin, etc. The vectors $|j, m\rangle$ can thus have several realizations. For example, in the spectroscopy of a partly filled shell ion with configuration $n\ell^N$, we have state vectors of type $|J, M\rangle \equiv |n\ell^N \tau SLJM\rangle$ in the Russell-Saunders coupling (here $j = J$ and $m = M$).

The vectors $|j, m\rangle$ can also serve for quantum information and quantum computation. By introducing the notation

$$k = j - m, \quad |k\rangle = |j,m\rangle, \quad d = 2j+1,$$

we get the vectors $|0\rangle$ (for $m = j$), $|1\rangle$ (for $m = j-1$), …, $|d-1\rangle$ (for $m = -j$). The vectors $|0\rangle$, $|1\rangle$, …, $|d-1\rangle$ are called qudits. Then, the basis b_s becomes

$$B_d = \{|k\rangle : k = 0, 1, \ldots, d-1\},$$

which is referred to as the computational basis in quantum information theory. In d dimensions, the most general qudit is written as

$$c_0|0\rangle + c_1|1\rangle + \ldots + c_{d-1}|d-1\rangle,$$

where $c_k \in \mathbf{C}$ for $k = 0, 1, \ldots, d-1$. The case $d = 2 \Leftrightarrow j = 1/2$ corresponds to qubits generated by $|0\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$ and $|1\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$. Note that the vectors $|\frac{1}{2}, \frac{1}{2}\rangle$ and $|\frac{1}{2}, -\frac{1}{2}\rangle$ are nothing but the spinorbitals α (for spin up) and β (for spin down), respectively, used in quantum chemistry.

In the rest of this paper, we shall use both the notation $|k\rangle$ familiar in (i) quantum information and quantum computation and (ii) the description of cyclic systems (for which $|d\rangle \equiv |0\rangle$, $|d+1\rangle \equiv |1\rangle$, $\ldots$) and equally well the notation $|j, m\rangle$ employed in (i) angular momentum theory (from a physical point of view or in its approach from the group SU_2), (ii) nuclear, atomic and molecular spectroscopy, and (iii) quantum chemistry. For $d = 2$, we shall play with the notation

$$\alpha = |\frac{1}{2}, \frac{1}{2}\rangle = |0\rangle$$

and

$$\beta = |\frac{1}{2}, -\frac{1}{2}\rangle = |1\rangle.$$

2.2 A noncanonical basis for SU_2

We now define the operator v_{0a} (a particular case of the unitary operator v_{ra} introduced in [21,22]) through

$$v_{0a}|j, m\rangle = (1 - \delta_{m,j})\, q^{(j-m)a}|j, m+1\rangle + \delta_{m,j}|j, -j\rangle$$

or equivalently

$$v_{0a}|k\rangle = q^{ka}|k-1\rangle,$$

where $a = 0, 1, \ldots, 2j = d-1$, $q = \exp(2\pi i/d)$, and $k-1$ should be understood modulo d (i.e., $|-1\rangle = |d-1\rangle$). It is evident that the operators j^2 and v_{0a} commute so that the (complete) set $\{j^2, v_{0a}\}$ constitutes an alternative to the set $\{j^2, j_z\}$ [5–7,18]. The matrix V_{0a} of the linear operator v_{0a} on the basis $b_s \equiv B_d$ reads

$$V_{0a} = \begin{pmatrix} 0 & q^a & 0 & \ldots & 0 \\ 0 & 0 & q^{2a} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & q^{2ja} \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

where the d $(= 2j+1)$ lines and d columns are labeled in the order $|0\rangle = |j, j\rangle, |1\rangle = |j, j-1\rangle, \ldots, |d-1\rangle = |j, -j\rangle$.

By introducing the Hermitian operator h via

$$h|j, m\rangle = \sqrt{(j+m)(j-m+1)}|j, m\rangle$$

or

$$h|k\rangle = \sqrt{(d-1-k)(k+1)}|k\rangle,$$

it can be shown that the three operators

$$j_+ = hv_{0a}, \quad j_- = v_{0a}^\dagger h, \quad j_z = \frac{1}{2}(h^2 - v_{0a}^\dagger h^2 v_{0a})$$

satisfy the commutation relations

$$[j_z, j_+] = +j_+, \quad [j_z, j_-] = -j_-, \quad [j_+, j_-] = 2j_z.$$

Therefore, j_+, j_- and j_z span the Lie algebra su_2 of SU_2 and the operators v_{0a} and h realize a polar decomposition of su_2 [5–7,18–19,21–22].

We may ask what are the analogues of the vectors $|j, m\rangle$ in the $\{j^2, v_{0a}\}$ scheme? In fact, the common eigenvectors of the commuting operators v_{0a} and j^2 are [18–19,21–22]

$$|a\alpha\rangle = \frac{1}{\sqrt{2j+1}} \sum_{m=-j}^{j} q^{(j+m)(j-m+1)a/2+(j+m)\alpha}|j, m\rangle$$

or alternatively

$$|a\alpha\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} q^{(d-k-1)(k+1)a/2-(k+1)\alpha}|k\rangle,$$

where α can take the values $\alpha = 0, 1, \ldots, 2j = d-1$. These vectors satisfy the eigenvalue equation

$$v_{0a}|a\alpha\rangle = q^{ja-\alpha}|a\alpha\rangle = q^{(d-1)a/2-\alpha}|a\alpha\rangle$$

that corresponds to a nondegenerate spectrum for the operator v_{0a}. For fixed j and a (with $2j \in \mathbf{N}$ and $a \in \mathbf{Z}_d$), the set

$$B_{0a} = \{|a\alpha\rangle : \alpha = 0, 1, \ldots, d-1\}$$

is an orthonormal basis for $\mathcal{E}(d)$, which is an alternative to the basis B_d.

3. APPLICATION TO QUANTUM INFORMATION

We are now in a position to discuss some results of interest for quantum information.

3.1 Weyl pairs

Besides the operator v_{0a}, it is interesting to define the unitary operator z through

$$z = (v_{00})^\dagger v_{01}.$$

This definition yields

$$z|j, m\rangle = q^{j-m}|j, m\rangle \Leftrightarrow z|k\rangle = q^k|k\rangle.$$

Therefore, we have the shift property

$$z|a\alpha\rangle = q^{-1}|a\alpha_1\rangle, \quad \alpha_1 = \alpha - 1.$$

Furthermore, we can show that v_{0a} is connected to z and $x = v_{00}$ by

$$v_{0a} = xz^a.$$

The two isospectral operators x (a shift operator when acting on $|j,m\rangle$ and a phase operator when acting on $|a\alpha\rangle$) and z (a phase operator when acting on $|j,m\rangle$ and a shift operator when acting on $|a\alpha\rangle$) are often called *shift* operator and *clock* operator, respectively, in quantum information and quantum computation. Note that for each of the operators x and z, the *shift* or *clock* character depends on which state, $|j,m\rangle = |k\rangle$ or $|a\alpha\rangle$, the operator acts. The operators v_{0a} and z satisfy

$$e^{-i\pi(d-1)a}\,(v_{0a})^d = z^d = 1, \quad v_{0a}z - qzv_{0a} = 0.$$

In other words, the unitary operators v_{0a} and z are cyclic (up to a phase factor for v_{0a} with $a \neq 0$) and do not commute. The pair (x,z) corresponding to $a=0$ is a Weyl pair in the sense that $x^d = z^d = 1$ and $xz = qzx$. Such a pair can be used as an integrity basis for constructing the Lie algebra of the unitary group U_d [24].

In the basis B_d, the d-dimensional matrices X and Z of the linear operators x and z are given by

$$X = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

and

$$Z = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & q & 0 & \ldots & 0 \\ 0 & 0 & q^2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & q^{d-1} \end{pmatrix}.$$

For $d=2 \Leftrightarrow j=1/2$ ($\Rightarrow q=-1$), the matrices V_{00}, V_{01}, and Z are connected to the Pauli matrices σ_x, σ_y, and σ_z via

$$V_{00} = X = \sigma_x, \quad V_{01} = XZ = -i\sigma_y, \quad Z = \sigma_z.$$

For d arbitrary, the pair ($X = V_{00}$, $Z = V_{00}^{\dagger}V_{01}$) are basic ingredients for generating generalized Pauli matrices and a generalized Pauli group P_d of interest in quantum computation for quantum correcting codes [24,25].

3.2 Mutually unbiased bases

We now give two applications to MUBs.

First, in the general case where d is arbitrary, we can check that

$$|\langle k|a\alpha\rangle| = \frac{1}{\sqrt{d}}, \quad k,a,\alpha \in \mathbf{Z}_d$$

and

$$|\langle 0\alpha|1\beta\rangle| = \frac{1}{\sqrt{d}}, \quad \alpha,\beta \in \mathbf{Z}_d,$$

where we use $\langle\ |\ \rangle$ to denote the inner product in $\mathcal{E}(d)$. In the terminology of quantum information, the latter two equations mean that the bases B_d, B_{00}, and B_{01} are three MUBs. This result is in agreement with the one according to which there exist at least three MUBs in arbitrary dimension, see for instance [17].

Second, in the special case where $d=p$ is a prime number, the latter equation can be extended to

$$|\langle a\alpha|b\beta\rangle| = \delta_{\alpha,\beta}\delta_{a,b} + \frac{1}{\sqrt{p}}(1-\delta_{a,b}), \quad a,b,\alpha,\beta \in \mathbf{Z}_d.$$

The proof follows from the use of generalized quadratic Gauss sums [21]. Hence, the bases $B_{00}, B_{01}, \ldots, B_{0p-1}$, and B_p constitute a (maximal) set of $p+1$ MUBs. This result is a particular case of the one according to which there exist a maximal set of $d+1$ MUBs when d is the power of a prime number [8–17].

We continue with some typical examples of interest for quantum information and quantum computation.

4. EXAMPLES

4.1 The case $d=2$

In this case, relevant for a spin $j=1/2$ or for a qubit, we have $q=-1$ and $a,\alpha \in \mathbf{Z}_2$. The matrices of the operators v_{0a} are

$$V_{00} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x, \quad V_{01} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_y.$$

The $d+1=3$ MUBs B_2, B_{00}, and B_{01} are the following.

The B_2 basis:

$$|0\rangle, \quad |1\rangle.$$

The B_{00} basis:

$$|00\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right), \quad |01\rangle = \frac{1}{\sqrt{2}}\left(-|0\rangle + |1\rangle\right).$$

The B_{01} basis:

$$|10\rangle = \frac{1}{\sqrt{2}}\left(i|0\rangle + |1\rangle\right), \quad |11\rangle = \frac{1}{\sqrt{2}}\left(-i|0\rangle + |1\rangle\right).$$

or, by using the spinorbitals α and β, we get

The B_2 basis:

$$\alpha, \quad \beta.$$

The B_{00} basis:

$$|00\rangle = \frac{1}{\sqrt{2}}\left(\alpha + \beta\right), \quad |01\rangle = -\frac{1}{\sqrt{2}}\left(\alpha - \beta\right).$$

The B_{01} basis:

$$|10\rangle = i\frac{1}{\sqrt{2}}\left(\alpha - i\beta\right), \quad |11\rangle = -i\frac{1}{\sqrt{2}}\left(\alpha + i\beta\right).$$

In terms of eigenvectors of the matrices V_{0a}, we must replace the state vectors $|a\alpha\rangle$ by column vectors. This leads to

The B_2 basis:

$$\alpha \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \beta \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The B_{00} basis:

$$|00\rangle \to \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |01\rangle \to -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The B_{01} basis:

$$|10\rangle \to i\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad |11\rangle \to -i\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

4.2 The case $d = 3$

This case corresponds to a spin $j = 1$ or to a qutrit. Here, we have $q = \exp(2\pi i/3)$ and $a, \alpha \in \mathbf{Z}_3$. The matrices of the operators v_{0a} are

$$V_{00} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$
$$V_{01} = \begin{pmatrix} 0 & q & 0 \\ 0 & 0 & q^2 \\ 1 & 0 & 0 \end{pmatrix}$$
$$V_{02} = \begin{pmatrix} 0 & q^2 & 0 \\ 0 & 0 & q \\ 1 & 0 & 0 \end{pmatrix}.$$

The d+1 = 4 MUBs B_3, B_{00}, B_{01}, and B_{02} are the following.

The B_3 basis:

$$|0\rangle, \quad |1\rangle, \quad |2\rangle.$$

The B_{00} basis:

$$|00\rangle = \frac{1}{\sqrt{3}} (|0\rangle + |1\rangle + |2\rangle)$$
$$|01\rangle = \frac{1}{\sqrt{3}} \left(q^2|0\rangle + q|1\rangle + |2\rangle\right)$$
$$|02\rangle = \frac{1}{\sqrt{3}} \left(q|0\rangle + q^2|1\rangle + |2\rangle\right).$$

The B_{01} basis:

$$|10\rangle = \frac{1}{\sqrt{3}} (q|0\rangle + q|1\rangle + |2\rangle)$$
$$|11\rangle = \frac{1}{\sqrt{3}} \left(|0\rangle + q^2|1\rangle + |2\rangle\right)$$
$$|12\rangle = \frac{1}{\sqrt{3}} \left(q^2|0\rangle + |1\rangle + |2\rangle\right).$$

The B_{02} basis:

$$|20\rangle = \frac{1}{\sqrt{3}} \left(q^2|0\rangle + q^2|1\rangle + |2\rangle\right)$$
$$|21\rangle = \frac{1}{\sqrt{3}} (q|0\rangle + |1\rangle + |2\rangle)$$
$$|22\rangle = \frac{1}{\sqrt{3}} (|0\rangle + q|1\rangle + |2\rangle).$$

This can be transcribed in terms of column vectors as follows.

The B_3 basis:

$$|0\rangle \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |1\rangle \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |2\rangle \to \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The B_{00} basis:

$$|00\rangle \to \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; |01\rangle \to \frac{1}{\sqrt{3}} \begin{pmatrix} q^2 \\ q \\ 1 \end{pmatrix}, \; |02\rangle \to \frac{1}{\sqrt{3}} \begin{pmatrix} q \\ q^2 \\ 1 \end{pmatrix}.$$

The B_{01} basis:

$$|10\rangle \to \frac{1}{\sqrt{3}} \begin{pmatrix} q \\ q \\ 1 \end{pmatrix}, \; |11\rangle \to \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ q^2 \\ 1 \end{pmatrix}, \; |12\rangle \to \frac{1}{\sqrt{3}} \begin{pmatrix} q^2 \\ 1 \\ 1 \end{pmatrix}.$$

The B_{02} basis:

$$|20\rangle \to \frac{1}{\sqrt{3}} \begin{pmatrix} q^2 \\ q^2 \\ 1 \end{pmatrix}, \; |21\rangle \to \frac{1}{\sqrt{3}} \begin{pmatrix} q \\ 1 \\ 1 \end{pmatrix}, \; |22\rangle \to \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ q \\ 1 \end{pmatrix}.$$

4.3 The case $d = 4$

This case corresponds to a spin $j = 3/2$. Here, we have $q = i$ and $a, \alpha \in \mathbf{Z}_4$. The five bases B_4, B_{00}, B_{01}, B_{02}, and B_{03} do not constitute a system of MUBs ($d = 4$ is not a prime number). Nevertheless, it is possible to find $d + 1 = 5$ MUBs because $d = 2^2$ is the power of a prime number. This can be achieved by replacing the space $\mathcal{E}(4)$ spanned by $\{|3/2, m\rangle : m = 3/2, 1/2, -1/2, -3/2\}$ by the tensor product space $\mathcal{E}(2) \otimes \mathcal{E}(2)$ spanned by the canonical (or computational) basis

$$\{\alpha \otimes \alpha, \alpha \otimes \beta, \beta \otimes \alpha, \beta \otimes \beta\}.$$

The space $\mathcal{E}(2) \otimes \mathcal{E}(2)$ is associated with the coupling of two spin angular momenta $j_1 = 1/2$ and $j_2 = 1/2$ or two qubits (in the vector $u \otimes v$, u and v correspond to j_1 and j_2, respectively). An alternative basis for $\mathcal{E}(2) \otimes \mathcal{E}(2)$ is the SU_2 adapted basis

$$\{\alpha \otimes \alpha, \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha), \beta \otimes \beta, \frac{1}{2}(\alpha \otimes \beta - \beta \otimes \alpha)\},$$

the vectors of which are well-known in the treatment of spin systems. Such a basis corresponds to the decomposition

$$(1/2) \otimes (1/2) = (1) \oplus (0)$$

in terms of irreducible representation classes of SU_2. In the SU_2 adapted basis, the first three vectors are symmetric under the interchange $1 \leftrightarrow 2$ and describe a total angular momentum $J = 1$ while the last one is antisymmetric and corresponds to $J = 0$. It should be observed that the SU_2 adapted basis illustrates a connection between the special unitary group SU_2 and the permutation group S_2 (a particular case of the Schur-Weyl duality theorem between

irreducible representation classes of the linear group GL_d and the symmetric group S_n).

In addition to the computational or canonical basis and the SU_2 adapted basis, it is possible to find other bases of $\mathcal{E}(2) \otimes \mathcal{E}(2)$ which are mutually unbiased. As a matter of fact, $d = 4$ MUBs, besides the computational basis $\{\alpha\otimes\alpha, \alpha\otimes\beta, \beta\otimes\alpha, \beta\otimes\beta\}$, correspond to the eigenvectors

$$|ab\alpha\beta\rangle = |a\alpha\rangle \otimes |b\beta\rangle$$

of the operators $w_{ab} = v_{0a} \otimes v_{0b}$ (the vectors $|a\alpha\rangle$ and $|b\beta\rangle$ refer to the two spaces $\mathcal{E}(2)$). As a result, we have the $d + 1 = 5$ following MUBs where $\lambda = (1 - i)/2$ and $\mu = (1 + i)/2$.

The canonical basis:

$$\alpha \otimes \alpha, \quad \alpha \otimes \beta, \quad \beta \otimes \alpha, \quad \beta \otimes \beta$$

or in column vectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The w_{00} basis:

$$|0000\rangle = \frac{1}{2}(\alpha\otimes\alpha + \alpha\otimes\beta + \beta\otimes\alpha + \beta\otimes\beta)$$
$$|0001\rangle = \frac{1}{2}(\alpha\otimes\alpha - \alpha\otimes\beta + \beta\otimes\alpha - \beta\otimes\beta)$$
$$|0010\rangle = \frac{1}{2}(\alpha\otimes\alpha + \alpha\otimes\beta - \beta\otimes\alpha - \beta\otimes\beta)$$
$$|0011\rangle = \frac{1}{2}(\alpha\otimes\alpha - \alpha\otimes\beta - \beta\otimes\alpha + \beta\otimes\beta)$$

or in column vectors

$$\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}.$$

The w_{11} basis:

$$|1100\rangle = \frac{1}{2}(\alpha\otimes\alpha + i\alpha\otimes\beta + i\beta\otimes\alpha - \beta\otimes\beta)$$
$$|1101\rangle = \frac{1}{2}(\alpha\otimes\alpha - i\alpha\otimes\beta + i\beta\otimes\alpha + \beta\otimes\beta)$$
$$|1110\rangle = \frac{1}{2}(\alpha\otimes\alpha + i\alpha\otimes\beta - i\beta\otimes\alpha + \beta\otimes\beta)$$
$$|1111\rangle = \frac{1}{2}(\alpha\otimes\alpha - i\alpha\otimes\beta - i\beta\otimes\alpha - \beta\otimes\beta)$$

or in column vectors

$$\frac{1}{2}\begin{pmatrix} 1 \\ i \\ i \\ -1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 \\ -i \\ i \\ 1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 \\ i \\ -i \\ 1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 \\ -i \\ -i \\ -1 \end{pmatrix}.$$

The w_{01} basis:

$$\lambda|0100\rangle + \mu|0111\rangle = \frac{1}{2}(\alpha\otimes\alpha + \alpha\otimes\beta - i\beta\otimes\alpha + i\beta\otimes\beta)$$
$$\mu|0100\rangle + \lambda|0111\rangle = \frac{1}{2}(\alpha\otimes\alpha - \alpha\otimes\beta + i\beta\otimes\alpha + i\beta\otimes\beta)$$
$$\lambda|0101\rangle + \mu|0110\rangle = \frac{1}{2}(\alpha\otimes\alpha - \alpha\otimes\beta - i\beta\otimes\alpha - i\beta\otimes\beta)$$
$$\mu|0101\rangle + \lambda|0110\rangle = \frac{1}{2}(\alpha\otimes\alpha + \alpha\otimes\beta + i\beta\otimes\alpha - i\beta\otimes\beta)$$

or in column vectors

$$\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -i \\ i \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ i \\ i \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -i \\ -i \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ i \\ -i \end{pmatrix}.$$

The w_{10} basis:

$$\lambda|1000\rangle + \mu|1011\rangle = \frac{1}{2}(\alpha\otimes\alpha - i\alpha\otimes\beta + \beta\otimes\alpha + i\beta\otimes\beta)$$
$$\mu|1000\rangle + \lambda|1011\rangle = \frac{1}{2}(\alpha\otimes\alpha + i\alpha\otimes\beta - \beta\otimes\alpha + i\beta\otimes\beta)$$
$$\lambda|1001\rangle + \mu|1010\rangle = \frac{1}{2}(\alpha\otimes\alpha + i\alpha\otimes\beta + \beta\otimes\alpha - i\beta\otimes\beta)$$
$$\mu|1001\rangle + \lambda|1010\rangle = \frac{1}{2}(\alpha\otimes\alpha - i\alpha\otimes\beta - \beta\otimes\alpha - i\beta\otimes\beta)$$

or in column vectors

$$\frac{1}{2}\begin{pmatrix} 1 \\ -i \\ 1 \\ i \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 \\ i \\ -1 \\ i \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 \\ i \\ 1 \\ -i \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 \\ -i \\ -1 \\ -i \end{pmatrix}.$$

It is to be noted that the vectors of the w_{00} and w_{11} bases are not intricated (i.e., each vector is the direct product of two vectors) while the vectors of the w_{01} and w_{10} bases are intricated (i.e., each vector is not the direct product of two vectors).

To be more precise, the degree of intrication of the state vectors for the bases w_{00}, w_{11}, w_{01}, and w_{10} can be determined in the following way. In arbitrary dimension d, let

$$|\Phi\rangle = \sum_{k=0}^{d-1}\sum_{l=0}^{d-1} a_{kl}|k\rangle \otimes |l\rangle$$

be a double qudit state vector. Then, it can be shown that the determinant of the $d \times d$ matrix $A = (a_{kl})$ satisfies

$$0 \le |\det A| \le \frac{1}{\sqrt{d^d}}$$

as proved in the Albouy thesis [26]. The case $\det A = 0$ corresponds to the absence of *global* intrication while the case

$$|\det A| = \frac{1}{\sqrt{d^d}}$$

corresponds to a maximal intrication. As an illustration, we obtain that all the state vectors for w_{00} and w_{11} are not intricated and that all the state vectors for w_{01} and w_{10} are maximally intricated.

5. CONCLUSION

We presented some results concerning bases of interest for spin systems and quantum information. It should be noted that the bases derived in the present work are adapted to cyclic symmetry and can be used for describing finite or infinite cyclic systems. As an extension of this paper and of its companion paper [24], it would be interesting to examine when d is a composite number how the Weyl pair (X, Z) and the corresponding generalized Pauli group P_d are connected to the Weyl pairs and the corresponding generalized Pauli group for each of the composite dimension.

We close with a comment on the group P_d. Such a group can be generated from the Weyl pair (X, Z). Indeed, the group P_d is a finite group of order d^3 with elements $q^a X^b Z^c$ (where $a, b, c \in \mathbf{Z}_d$) and matrix multiplication for group law. It is a subgroup of the unitary group U_d. The normaliser of P_d in U_d is a Clifford-type group in d dimensions noted C_d. More precisely, C_d is the set $\{U \in \mathrm{U}_d | U P_d U^\dagger = P_d\}$ endowed with matrix multiplication. The Pauli group P_d as well as any other invariant subgroup of C_d can be used for stabilizing errors in quantum computing. These concepts are very important in the case of n-qubit systems (corresponding to $d = 2^n$). These matters, in connection with the decomposition of P_{2^n} in terms of P_2, are the object of a future work.

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